SUBSPACE ARRANGEMENTS DEFINED BY PRODUCTS OF LINEAR FORMS

ANDERS BJÖRNER, IRENA PEEVA, AND JESSICA SIDMAN

ABSTRACT. We consider the vanishing ideal of an arrangement of linear subspaces in a vector space and investigate when this ideal can be generated by products of linear forms. We introduce a combinatorial construction (blocker duality) which yields such generators in cases with a lot of combinatorial structure, and we present the examples that motivated our work. We give a construction which produces all elements of this type in the vanishing ideal of the arrangement. This leads to an algorithm for deciding if the ideal is generated by products of linear forms. We also consider generic arrangements of points in $\mathbf{P}^2$ and lines in $\mathbf{P}^3$.

1. INTRODUCTION

Throughout the paper k is an infinite field. We consider an arrangement $\mathcal{A}$ of r linear subspaces in k^n; we assume that none of the subspaces contains another. Let $I_1, \ldots, I_r$ be the linear ideals in $k[x_1, \ldots, x_n]$ that are the defining ideals of the subspaces in $\mathcal{A}$. Denote by $V_{\mathcal{A}}$ the union of the subspaces in $\mathcal{A}$. The vanishing ideal of $V_{\mathcal{A}}$ is the reduced ideal

$$I_{\mathcal{A}} = I_1 \cap \cdots \cap I_r.$$

The ideal defining a subspace arrangement arises in connection with topics as diverse as independence numbers of graphs and graph coloring (see [LL1], [LL2], [Lo], [dL], [Do]), invariant theory [De], and symmetric function theory [Ha].

When $\mathcal{A}$ is an arrangement of hyperplanes its vanishing ideal $I_{\mathcal{A}}$ is a very simple object – a principal ideal generated by the product of linear forms that define the hyperplanes. In general, the ideal $I_{\mathcal{A}}$ is generated by products of linear forms up to a radical, since $\mathrm{rad}(I_1 \cdots I_r) = \mathrm{rad}(I_1) \cap \cdots \cap \mathrm{rad}(I_r) = I_1 \cap \cdots \cap I_r = I_{\mathcal{A}}$, but it is difficult to construct a nice system of generators of $I_{\mathcal{A}}$ itself. Geometrically, finding generators of $I_{\mathcal{A}}$ is related to detecting low-degree

hypersurfaces intersecting in $V_{\mathcal{A}}$. We will present examples where the ideal $I_{\mathcal{A}}$ is generated by products of linear forms in many cases in which $\mathcal{A}$ has a great deal of combinatorial structure.

We say that an ideal is *pl-generated* if it is generated by products of linear forms. In this paper, we study combinatorial properties of $\mathcal{A}$ that are related to $I_{\mathcal{A}}$ being pl-generated. We present the combinatorial point of view in §3 and the ideal-theoretic point of view in §4. The last section is entirely different in flavor: in §5 we study when $I_{\mathcal{A}}$ is pl-generated for arrangements in $\mathbf{P}^2$ and $\mathbf{P}^3$.

In §3 we introduce the notion of blocker duality, a combinatorial operation which, given a subspace arrangement $\mathcal{A}$ and an embedding $\mathcal{A} \subseteq \mathcal{H}$ into a hyperplane arrangement $\mathcal{H}$, produces another "dual" subspace arrangement $\mathcal{A}^*$. It is not in general true that $\mathcal{A}^{**} = \mathcal{A}$, only that $V_{\mathcal{A}^{**}} \supseteq V_{\mathcal{A}}$.

We provide an overview of the examples that motivated this construction in §3.1. In §3.2 we define blocker duality and demonstrate its basic properties. In §3.3 we use blocker duality to define a pl-generated ideal $B_{\mathcal{A}, \mathcal{H}}$ which is contained in $I_{\mathcal{A}}$. Over an algebraically closed field we show that $\mathcal{A} = \mathcal{A}^{**}$ if and only if $\mathrm{rad}(B_{\mathcal{A}, \mathcal{H}}) = I_{\mathcal{A}}$.

The stronger property, that $B_{\mathcal{A}, \mathcal{H}} = I_{\mathcal{A}}$, holds for our motivating examples, as well as for some other fundamental examples which we discuss in §3.4. One would like to determine a combinatorial property of $\mathcal{A}$, viewed as an antichain in the intersection lattice $L_{\mathcal{H}}$ of $\mathcal{H}$, that makes it possible to detect if $B_{\mathcal{A}, \mathcal{H}}$ is radical and explains the examples.

In §4, we depart from the beautiful examples where blocker duality works and we consider the more general situation where the blocker ideal $B_{\mathcal{A}, \mathcal{H}}$ may fail to be equal to $I_{\mathcal{A}}$. This could be caused by the following two problems:

(1) $B_{\mathcal{A}, \mathcal{H}}$ may fail to capture all products of linear forms in $\mathcal{H}$ that are contained in $I_{\mathcal{A}}$.

(2) It might not be possible to generate enough products of linear forms using only linear forms from $\mathcal{H}$.

In §4.1 we solve the first problem by introducing the ideal $F_{\mathcal{A}, \mathcal{H}}$ which is larger than $B_{\mathcal{A}, \mathcal{H}}$. It is constructed combinatorially, but it is also a natural algebraic object: $F_{\mathcal{A}, \mathcal{H}}$ is the largest ideal inside $I_{\mathcal{A}}$ that is generated by products of linear forms in $\mathcal{H}$.

In §4.2 we solve the second problem. We prove that any given embedding $\mathcal{A} \subseteq \mathcal{H}$ can be enlarged to an embedding $\mathcal{A} \subseteq \tilde{\mathcal{H}}$ so that $F_{\mathcal{A}, \tilde{\mathcal{H}}}$ is the ideal generated by all products of linear forms inside $I_{\mathcal{A}}$. In particular, Theorem 4.2.4 shows that if $\mathcal{A}$ has the pl-property then

a system of generators of $I_{\mathcal{A}}$ that are products of linear forms can be constructed by a combinatorial procedure. As an immediate consequence we obtain Algorithm 4.2.5, which makes it possible to check by computer whether a given ideal $I_{\mathcal{A}}$ is pl-generated. For a related result see Proposition 1.1 in [LL2].

In §5 we will see that the ideals of generic subspace arrangements often fail to be pl-generated. We study arrangements of points in $\mathbf{P}^2$ in §5.1 and arrangements of lines in $\mathbf{P}^3$ in §5.2. Propositions 5.1.4 and 5.2.1 show that the ideals of generic arrangements of points in $\mathbf{P}^2$ (respectively lines in $\mathbf{P}^3$) are not pl-generated when the number of subspaces is large. However, in both cases generic arrangements are scheme-theoretically cut out by products of linear forms. This is true for any arrangement of pairwise disjoint subspaces; it is easy to see that the union of any disjoint subschemes of projective space cut out by ideals $I_1, \ldots, I_r$ is scheme-theoretically defined by $I_1 \cdots I_r$. By contrast, Proposition 5.2 shows that there exist arrangements of lines in $\mathbf{P}^3$ that are not scheme-theoretically cut out by any pl-generated ideal.

2. Notation and conventions

We begin by briefly recalling a few basic definitions. If x, y are elements of a geometric lattice L then $x \wedge y$ is their *meet*, or greatest lower bound in L, and $x \vee y$ is their *join*, or least upper bound in L. The least element of L is denoted by $\hat{0}$, and the greatest by $\hat{1}$. A set $A \subseteq L$ is an *antichain* if $\hat{0} \notin A \neq \emptyset$ and the elements of A are pairwise incomparable with respect to the partial ordering in L.

We say that a subspace arrangement $\mathcal{A}$ is *embedded* in a hyperplane arrangement $\mathcal{H}$ if each $X \in \mathcal{A}$ is the intersection of some of the hyperplanes from $\mathcal{H}$. Throughout the paper, $\mathcal{A} \subseteq \mathcal{H}$ is an embedding into a central hyperplane arrangement $\mathcal{H}$ with intersection lattice $L_{\mathcal{H}}$. Denote by $\ell_1, \ldots, \ell_p$ the linear forms defining the hyperplanes in $\mathcal{H}$. We think of $L_{\mathcal{H}}$ as the geometric lattice with atoms $\ell_1, \ldots, \ell_p$. Denote

by $V_1, \ldots, V_r$ the elements in $L_{\mathcal{H}}$ that correspond to the subspaces in $\mathcal{A}$. The set $A_{\mathcal{A}, \mathcal{H}} = \{V_1, \ldots, V_r\}$ is an antichain in the lattice $L_{\mathcal{H}}$.

For a comprehensive introduction to general notions related to hyperplane arrangements and subspace arrangements see [OT] and [Bj], respectively. For the matroid and geometric lattice point of view see [Ox].

For simplicity, we assume that the field k is infinite. If k is finite then the ideal $I_{\mathcal{A}}$ should be defined as the intersection of ideals I_i generated by linear forms such that the Krull dimension of $k[x_1, \ldots, x_n]/I_i$ is the dimension of the corresponding vector subspace of k^n. However, this may be strictly contained in the ideal of all polynomials which vanish on the finitely many points of the subspaces.

3. BLOCKER DUALITY

In this section we define the blocker dual of a subspace arrangement embedded in a hyperplane arrangement and discuss properties of the associated blocker ideal. We begin and end the section by describing examples for which the blocker ideal is the radical ideal of an arrangement.

3.1. **Motivating examples.** The main motivation for the construction which we will describe in §3.2 comes from observing a beautiful duality between the subspaces of certain arrangements embedded in the braid arrangement and polynomials which generate their defining ideals. In order to discuss these examples we recall some basic facts about braid arrangements. (See [OT] and [Bj] for more details.)

The braid arrangement $\mathcal{H}_n$ is the arrangement of hyperplanes in k^n defined by the polynomial

$$\prod_{1 \leq i < j \leq n} (x_i - x_j).$$

We identify the intersection lattice of $\mathcal{H}_n$ with Π_n, the lattice of all partitions of $[n] = \{1, \ldots, n\}$, as follows. Given a partition π of $[n]$ into disjoint blocks, we define $i \equiv j$ if and only if i and j are in the same block of π and associate to π the linear subspace of k^n defined by the ideal $(x_i - x_j \mid 1 \leq i < j \leq n, \ i \equiv j)$.

The symmetric group S_n acts on the intersection lattice of $\mathcal{H}_n$ by permuting the subscripts of the coordinates of k^n. The orbits of this action are indexed by the shapes of partitions of the set $[n]$. We say that the *shape* of a partition π is the list of its block sizes arranged in non-increasing order. E.g., if π is the partition $\{\{1, 2, 3\}, \{4\}\}$ of $[4]$, then the shape of π is $(3, 1)$.

Let $\mathcal{A}_\lambda$ be the arrangement consisting of all subspaces corresponding to partitions of shape λ. The products

$$f_\pi = \prod_{\substack{i < j \\ i \equiv j}} (x_i - x_j)$$

for $\pi \in \Pi_n$ play an important role in what follows.

Example 3.1.1. In [LL1] Li and Li found an explicit system of generators for the ideals of orbit arrangements $\mathcal{A}_\lambda$ corresponding to "hook" shapes $\lambda = (m, 1, \ldots, 1)$. Namely, the vanishing ideal of the arrangement $\mathcal{A}_\lambda$ is

$$(f_\pi \mid \pi \text{ has } m - 1 \text{ blocks}).$$

Example 3.1.2. A result of Kleitman and Lovász in [Lo] describes a system of generators of the ideal of certain arrangements consisting of unions of orbit arrangements. Let

$$\mathcal{A}_m = \bigcup_{\lambda \text{ has } m-1 \text{ blocks}} \mathcal{A}_\lambda.$$

The defining ideal of $\mathcal{A}_m$ is

$$\big(f_\pi \mid \pi \text{ has shape } (m, 1, \ldots, 1)\big).$$

Note that the partitions indexing the subspaces of Example 3.1.1 index the generators of the ideal of the arrangement in Example 3.1.2, and vice versa. In the next section we define a combinatorial operation on antichains in a geometric lattice which captures this duality.

3.2. **The blocker.** We now define the notion of *blocker duality* motivated by Examples 3.1.1 and 3.1.2 and demonstrate its basic properties. This concept is purely combinatorial and for our purposes best discussed in the setting of geometric lattices. For more about the combinatorial properties of the blocker construction, see [Ma] and [BH].

Definition 3.2.1. Let A be an antichain in a geometric lattice L. The *blocker* of A is the antichain

$$A^* = \min \big\{ x \in L \mid a \wedge x \neq \hat{0} \text{ for every } a \in A \big\},$$

where $\min E$ denotes the set of minimal elements of a subset $E \subseteq L$.

Note that $A^* \neq \emptyset$, since $a \wedge \hat{1} = a \neq \hat{0}$ for all $a \in A$. As an example, let $A = \{\hat{1}\}$ and $B = \{\text{atoms}\}$. Then $A^* = B$ and $B^* = A$.

A partial order on the antichains in a geometric lattice L is defined as follows: we say that $A \leq B$ for two antichains if for each $b \in B$ there exists an $a \in A$ such that $a \leq b$. The proof of the following lemma is straightforward.

Lemma 3.2.2.
  (1) *If $A \leq B$, then $B^* \leq A^*$.*
  (2) $A^{**} \leq A$.

The following proposition describes the sense in which the $*$ operation on antichains is a reflexive duality operation. Note that the notion of reflexivity given by $*$ is somewhat weak: $A = A^{**}$ does not hold in general.

Proposition 3.2.3. *Let A be an antichain in a geometric lattice L. Then $A^* = A^{***}$.*

Proof. By Lemma 3.2.2 (2) we get that $A^{***} \leq A^*$. On the other hand, Lemma 3.2.2 (1) applied to $A^{**} \leq A$ yields $A^{***} \geq A^*$.   □

The definition of the blocker is designed to generalize the duality between Examples 3.1.1 and 3.1.2. Indeed, we have:

Example 3.2.4. Let $\lambda = (m, 1, \ldots, 1)$, and let A_λ be the antichain in Π_n of partitions of the set $[n]$ of shape λ. Then

$$A_\lambda^* = \{\pi \in \Pi_n \mid \pi \text{ has } m-1 \text{ blocks}\}$$

and $A_\lambda^{**} = A_\lambda$. The antichain A_λ corresponds to the orbit arrangement $\mathcal{A}_\lambda$ embedded in the braid arrangement $\mathcal{H}_n$, and A_λ^* corresponds to the arrangement

$$\mathcal{A}_m = \bigcup_{\lambda \text{ has } m-1 \text{ blocks}} \mathcal{A}_\lambda \,.$$

Remark 3.2.5. The blocker construction was originally introduced (in [EF] and other places) for the special case when L is the Boolean lattice of all subsets of a finite set V. In this case it is known that $A^{**} = A$ for all antichains A. See Example 3.4.2 for more about this.

The generalization of blockers to posets has also independently been considered by Matveev [Ma].

3.3. **The blocker ideal.** We now define the blocker ideal $B_\mathcal{A} = B_{\mathcal{A}, \mathcal{H}}$ of an arrangement $\mathcal{A}$, with respect to an embedding $\mathcal{A} \subseteq \mathcal{H}$, and show some of its most basic connections with the vanishing ideal $I_\mathcal{A}$.

Suppose that a subspace arrangement $\mathcal{A}$ is embedded in a hyperplane arrangement $\mathcal{H}$, i.e. $X \in L_\mathcal{H}$ for all $X \in \mathcal{A}$. Let $A_{\mathcal{A}, \mathcal{H}}$ denote $\mathcal{A}$ viewed as an antichain in $L_\mathcal{H}$ and $A_{\mathcal{A}, \mathcal{H}}^*$ denote its blocker dual. When confusion cannot arise we simplify notation by suppressing the reference to $\mathcal{H}$ and identifying the subspace arrangements embedded in $\mathcal{H}$ with the antichains contained in $L_\mathcal{H}$. Thus, we may speak directly of the blocker dual $\mathcal{A}^*$ of a subspace arrangement $\mathcal{A}$. Note that the operation

$\mathcal{A} \to \mathcal{A}^{**}$ defines a closure operation on subspace arrangements (with respect to $\mathcal{H}$), namely, by Lemma 3.2.2: $V_\mathcal{A} \subseteq V_{\mathcal{A}^{**}}$.

Given an arrangement of hyperplanes $\mathcal{H}$ in which the hyperplanes are defined by linear forms $\ell_1, \ldots, \ell_p$, we associate a product of linear forms to each element of of $L_\mathcal{H}$ as follows:

Definition 3.3.1. For $X \in L_\mathcal{H}$, define

$$Q_X = \prod_{\ell_i(X) \equiv 0} \ell_i.$$

Using this definition we define the blocker ideal of $\mathcal{A} \subseteq \mathcal{H}$:

Definition 3.3.2. The *blocker ideal* $B_{\mathcal{A}, \mathcal{H}}$ is

$$B_{\mathcal{A}, \mathcal{H}} = (Q_X \mid X \in \mathcal{A}^*).$$

The first part of the following proposition shows that that $Q_X \in I_\mathcal{A}$ for all $X \in \mathcal{A}^*$, hence

$$B_{\mathcal{A}, \mathcal{H}} \subseteq I_\mathcal{A}.$$

The third part shows that the blocker ideal cuts out $V_{\mathcal{A}^{**}}$ set-theoretically.

Let $\mathcal{A}$ be a subspace arrangement embedded in the hyperplane arrangement $\mathcal{H}$, and let $\mathcal{A}^*$ be the blocker dual arrangement. For $X \in L_\mathcal{H}$ let $\mathcal{H}/X = \{H \in \mathcal{H} \mid H \supseteq X\}$.

Proposition 3.3.3.
 (1) $V_\mathcal{A} \quad \subseteq \bigcap_{X \in \mathcal{A}^*} V_{\mathcal{H}/X}$
 (2) $V_{\mathcal{A}^*} \supseteq \bigcap_{X \in \mathcal{A}} V_{\mathcal{H}/X}$
 (3) $V_{\mathcal{A}^{**}} = \bigcap_{X \in \mathcal{A}^*} V_{\mathcal{H}/X}$

Proof. (1) Suppose that $z \in C \in \mathcal{A}$. For each $X \in \mathcal{A}^*$ there exists (by definition of the blocker) a hyperplane $H_X \in \mathcal{H}$ such that $z \in C \cup X \subseteq H_X$. Then, $z \in \cap_{X \in \mathcal{A}^*} H_X \subseteq \cap_{X \in \mathcal{A}^*} V_{\mathcal{H}/X}$.

(2) Suppose that $y \in V_{\mathcal{H}/X}$ for all $X \in \mathcal{A}$. So, for each $X \in \mathcal{A}$ there is a hyperplane $H_X \supseteq X$ such that $y \in H_X$. Let $C = \cap_{X \in \mathcal{A}} H_X$. Then $C \wedge X \neq \hat{0}$ for all $X \in \mathcal{A}$, and hence there exists some $\widetilde{C} \in \mathcal{A}^*$ such that $\widetilde{C} \leq C$. We have that $y \in C \subseteq \widetilde{C} \subseteq V_{\mathcal{A}^*}$.

(3) Using the preceding parts we have that

$$V_{\mathcal{A}^{**}} \subseteq \bigcap_{X \in \mathcal{A}^{***}} V_{\mathcal{H}/X} = \bigcap_{X \in \mathcal{A}^*} V_{\mathcal{H}/X} \subseteq V_{\mathcal{A}^{**}}.$$

$\square$

Theorem 3.3.4. *Over an algebraically closed field, the following properties hold:*
 (1) $\mathrm{rad}(B_{\mathcal{A}, \mathcal{H}}) = I_{\mathcal{A}^{**}}$,

(2) $\mathrm{rad}(B_{\mathcal{A},\mathcal{H}}) = I_{\mathcal{A}}$ *if and only if* $\mathcal{A}^{**} = \mathcal{A}$.

Proof. The first statement follows directly from Proposition 3.3.3(3) via the Hilbert Nullstellensatz. Thus, $\mathrm{rad}(B_{\mathcal{A},\mathcal{H}})$ defines $V_{\mathcal{A}^{**}}$ and so the second part follows. □

As we will see with Example 4.2.2, the property $\mathcal{A} = \mathcal{A}^{**}$ does not guarantee that $B_{\mathcal{A},\mathcal{H}}$ is a radical ideal.

3.4. **More examples.** The notion of blocker duality behaves well for several interesting subspace arrangements. Here we give examples having the property that $B_{\mathcal{A},\mathcal{H}} = I_{\mathcal{A}}$, assuming only that the field k is infinite.

Example 3.4.1. Suppose that $\mathcal{H} = \{H_1, \ldots, H_p\}$ is a central hyperplane arrangement with defining equation $\ell_1 \cdots \ell_p$. Then $X := \cap_{i=1}^{p} H_i$ is an element of $L_{\mathcal{H}}$ and is the only element of $\mathcal{H}^*$. Furthermore, $Q_X = \ell_1 \cdots \ell_p$. Hence, $B_{\mathcal{H},\mathcal{H}} = I_{\mathcal{H}}$.

Example 3.4.2. We say that $\mathcal{A}$ is a *coordinate subspace arrangement*, or a *Boolean* arrangement (see [Bj, §3.2]), in k^n if each subspace in $\mathcal{A}$ is an intersection of coordinate hyperplanes. Such an $\mathcal{A}$ has a natural embedding into the coordinate hyperplane arrangement $\mathcal{C}_n$ defined by the ideal $(x_1 \cdots x_n)$, whose intersection lattice is isomorphic to the Boolean lattice B_n of all subsets of $[n]$. The blocker duals of coordinate subspace arrangements have close connections with the Stanley-Reisner rings of simplicial complexes and a nice interpretation in terms of Alexander duality, as we now show.

Let us begin set-theoretically. An antichain A in B_n generates an abstract simplicial complex $\Delta_A = \{X \subseteq [n] \mid X \subseteq F \text{ for some } F \in A\}$. Conversely, $\max(\Delta)$ is an antichain for every simplicial complex Δ. Clearly,

$$\Delta_{\max(\Delta)} = \Delta \quad \text{and} \quad \max(\Delta_A) = A,$$

so antichains and simplicial complexes are interchangeable concepts here.

Let $X^c = [n] \setminus X$ for subsets $X \subseteq [n]$, and $A^c = \{X^c \mid X \in A\}$ for antichains A. The simplicial complex $\Delta^{\mathrm{dual}} = \{X^c \mid X \notin \Delta\}$ is known as the *(combinatorial) Alexander dual* of Δ. By the previous comments we may instead speak of the Alexander dual A^{dual} of an antichain A in B_n.

We know from Example 3.2.5 that $A^{**} = A$ for all antichains A in B_n. We also know that $(A^{\mathrm{dual}})^{\mathrm{dual}} = A$ and $A^{cc} = A$ for all A. These duality operations are related as follows

(1) $$A^{\mathrm{dual}} = A^{c*c} \quad \text{and} \quad A^* = A^{c\,(\mathrm{dual})\,c},$$

since the definitions show that

$$(2) \qquad X \in A^{(\text{dual})\,c} \iff X \in \min\left(B_n \setminus \Delta_A\right) \iff X \in A^{c*}.$$

Let

$$G \subseteq [n] \quad \leftrightarrow \quad S_G = \{(x_1, \ldots, x_n) \in k^n \mid x_i = 0 \text{ for all } i \notin G\}$$

be the chosen correspondence between subsets of $[n]$ and coordinate subspaces. If $\mathcal{A}$ is a coordinate subspace arrangement corresponding to an antichain A in B_n with simplicial complex Δ_A, then $I_{\mathcal{A}}$ is a monomial ideal. Namely, the ideal $I_{\mathcal{A}}$ is generated by the square-free products of variables $\prod_{i \in G} x_i$, for all minimal non-faces $G \notin \Delta_A$ (see [Bj, §11.1]). This is known as the *Stanley-Reisner ideal* of Δ_A.

Under the order-reversing isomorphism $B_n \leftrightarrow L_{\mathcal{C}_n}$ given by $G \leftrightarrow S_G$ we have that $A \leftrightarrow \mathcal{A}$ implies that $A^{c*c} \leftrightarrow \mathcal{A}^*$. Furthermore,

$$\prod_{i \in G} x_i = Q_{S_{G^c}}$$

for all $G \subseteq [n]$. Equation (2) shows that $A^{c*} = \min\left(B_n \setminus \Delta_A\right)$, from which follows that $I_{\mathcal{A}}$ is generated by all Q_{S_G} such that $G \in A^{c*c}$. That is, $I_{\mathcal{A}}$ is in fact the blocker ideal.

Hence, for all coordinate subspace arrangements: $B_{\mathcal{A}, \mathcal{C}_n} = I_{\mathcal{A}}$.

Example 3.4.3. The results of [LL1] and [Lo] show that the blocker ideals of $\mathcal{A}_\lambda$, for λ of hook shape, and of its *-dual in the braid arrangement, are the respective radical ideals, cf. Examples 3.1.1, 3.1.2 and 3.2.4. Orbit arrangements $\mathcal{A}_\lambda$ are themselves in general not blockers with respect to the braid arrangement. A procedure for computing their blocker duals, and hence their blocker ideals, is given in [BH]. We do not know of any description of their vanishing ideals for general non-hook shapes

The following table gives the blocker duals, and double duals, with respect to the braid arrangement, for all $\mathcal{A}_\lambda$ indexed by partitions λ of $n = 6$ that are not of hook shape. Here $\ll m \gg$ denotes the union of all orbit arrangements for partitions with m blocks, as in Example 3.1.2.

$\mathcal{A}_\lambda$	$\mathcal{A}_\lambda^*$	$\mathcal{A}_\lambda^{**}$
$(4,2)$	$(2,2,2) \cup (3,1,1,1)$	$(4,2) \cup (5,1)$
$(3,3)$	$(3,1,1,1)$	$\ll 2 \gg$
$(3,2,1)$	$(3,3) \cup (4,1,1)$	$(3,2,1) \cup (4,1,1)$
$(2,2,2)$	$(4,1,1)$	$\ll 3 \gg$
$(2,2,1,1)$	$(5,1)$	$\ll 4 \gg$

Note that the arrangements $\mathcal{A}_{(2,2,2)} \cup \mathcal{A}_{(3,1,1,1)}$ and $\mathcal{A}_{(4,2)} \cup \mathcal{A}_{(5,1)}$ are blocker dual to each other, as are the arrangements $\mathcal{A}_{(3,3)} \cup \mathcal{A}_{(4,1,1)}$ and $\mathcal{A}_{(3,2,1)} \cup \mathcal{A}_{(4,1,1)}$. Using MACAULAY 2 [GS] we computed the ideals of these four arrangements and compared them to the respective blocker ideals. Working over the field $\mathbb{Q}$ we found that the blocker ideal in each case equals the vanishing ideal.

Example 3.4.4. Let E be a d-dimensional vector space over the field k. Given positive integers m and n and a function $f : [m] \to [n]$ let

$$W_f = \{(x_1, \ldots, x_n, x_{f(1)}, \ldots, x_{f(m)}) \mid x_i \in E\}.$$

This is an nd-dimensional linear subspace of E^{n+m}. Letting f range over all such functions, define the *polygraph arrangement*

$$\mathcal{Z}_E(n, m) = \{W_f \mid f : [m] \to [n]\}.$$

Such arrangements were introduced by M. Haiman in [Ha], and for $E = \mathbb{C}^2$ they play a crucial role in his proof of the $n!$ conjecture. They were further investigated from a combinatorial point of view in [Hu].

Now let $d = 1$, and consider the vanishing ideal $I_{\mathcal{Z}_k(n,m)}$ in the polynomial ring $k[x_1, \ldots, x_n, a_1, \ldots, a_m]$. Haiman [Ha, p. 966] shows that the ideal $I_{\mathcal{Z}_k(n,m)}$ is generated by

$$q_i = \prod_{j \in [n]} (x_j - a_i), \quad i \in [m].$$

This implies that $I_{\mathcal{Z}_k(n,m)}$ is the blocker ideal of $\mathcal{Z}_k(n, m)$ with respect to its embedding into the "bipartite braid arrangement" $\mathcal{H}(n, m) = \{x_j - a_i \mid j \in [n], i \in [m]\}$, as we now show.

Let P and Q be disjoint sets of cardinalities $|P| = m$ and $|Q| = n$, and let $\Pi_{P \cup Q}$ denote the lattice of all partitions of the set $P \cup Q$. Let $\Pi^\circ_{P,Q}$ denote the lattice that is join-generated within $\Pi_{P \cup Q}$ by all rank one partitions (atoms) whose only non-singleton block is of type $\{p, q\}$ with $p \in P$ and $q \in Q$. So, $\pi \in \Pi^\circ_{P,Q}$ if and only if every block of π either is a singleton or else intersects both P and Q.

The isomorphism of the intersection lattice of the braid arrangement $\mathcal{H}_{n+m}$ with $\Pi_{P \cup Q}$ (see §3.1) clearly restricts to an isomorphism $L_{\mathcal{H}(n,m)} \cong \Pi^\circ_{P,Q}$. Hence, we can compute the blocker dual of $\mathcal{Z}_k(n, m)$ within $\Pi^\circ_{P,Q}$.

The antichain in $\Pi^\circ_{P,Q}$ that corresponds to $\mathcal{Z}_k(n, m)$ under the stated isomorphism is the antichain $A_{n,m}$ of all partitions in $\Pi^\circ_{P,Q}$ for which each block contains *exactly one* element from Q (note that consequently each $\pi \in A_{n,m}$ contains exactly n blocks). Via combinatorial reasoning

one sees that

$$A^*_{n,m} = \{\pi \in \Pi^\circ_{P,Q} \mid \pi \text{ has a unique non-singleton block } Q \cup \{p_i\},$$
$$\text{for some } i \in [m]\}$$

$$A^{**}_{n,m} = A_{n,m}.$$

Hence, the generator set of the blocker ideal $B_{\mathcal{A},\mathcal{H}}$, for $\mathcal{A} = \mathcal{Z}_k(n,m)$ embedded in $\mathcal{H} = \mathcal{H}(n,m)$, is precisely the set of polynomials $\{q_i\}_{i \in [m]}$ defined above. In other words, $B_{\mathcal{A},\mathcal{H}} = I_{\mathcal{A}}$ for polygraph arrangements in case $d = 1$.

The situation becomes more complicated if $d > 1$. The combinatorics stays the same, but the algebra gets more involved. Haiman [Ha, §4.6, eq. (96)] gives a set of generators for the special case of $n = d = 2$, but states [Ha, p. 967] that "at present, we do not have a good conjecture as to a set of generators for the full ideal [for $d = 2$] in general".

4. PRODUCTS OF LINEAR FORMS INSIDE $I_{\mathcal{A}}$

In this section we show that the ideal generated by all products of linear forms that vanish on an arrangement $\mathcal{A}$ can be constructed by a simple algorithmic procedure. Given an embedding $\mathcal{A} \subset \mathcal{H}$, we construct the ideal $F_{\mathcal{A},\mathcal{H}}$ generated by all products of linear forms defining elements of $\mathcal{H}$. We then show how to generate an embedding of $\mathcal{A}$ into a hyperplane arrangement so that the ideal $F_{\mathcal{A},\mathcal{H}}$ is as large as possible.

4.1. **The $\mathcal{H}$-product ideal.** Example 4.1.2 shows that the blocker ideal $B_{\mathcal{A},\mathcal{H}}$ may fail to be equal to $I_{\mathcal{A}}$, as the blocker construction may not detect all products of linear forms in $I_{\mathcal{A}}$. We introduce the $\mathcal{H}$-*product ideal* $F_{\mathcal{A},\mathcal{H}}$, which corrects for this failure.

Definition 4.1.1. The $\mathcal{H}$-*product ideal* is the ideal

$$F_{\mathcal{A},\mathcal{H}} = \left(\ell_1 \cdots \ell_q \quad \mid \quad \ell_j \in L_{\mathcal{H}}, \text{ and for all } 1 \leq i \leq r \right.$$
$$\left. \text{there exists } 1 \leq j \leq q \text{ such that } \ell_j \in I_i \right).$$

Clearly, $B_{\mathcal{A},\mathcal{H}} \subseteq F_{\mathcal{A},\mathcal{H}} \subseteq I_{\mathcal{A}}$, and the first two ideals can be computed combinatorially given $L_{\mathcal{H}}$ and the antichain $A_{\mathcal{A},\mathcal{H}}$. We will show that a strict inclusion $B_{\mathcal{A},\mathcal{H}} \subset F_{\mathcal{A},\mathcal{H}}$ is possible. That strict inclusion $F_{\mathcal{A},\mathcal{H}} \subset I_{\mathcal{A}}$ is possible can be seen from Example 4.2.2.

Example 4.1.2. Let $\mathcal{H} = \mathcal{H}_3$ be the braid arrangement with hyperplanes $\{x_1 = x_2\}$, $\{x_1 = x_3\}$, $\{x_2 = x_3\}$. Consider the subspace

arrangement $\mathcal{A}$ with subspaces $\{x_1 = x_2\}$, $\{x_1 = x_3\}$. Clearly, $\mathcal{A} \subseteq \mathcal{H}$. Both ideals $B_{\mathcal{A},\mathcal{H}}$ and $F_{\mathcal{A},\mathcal{H}}$ are principal, but they are different, and furthermore their radicals are different as well. We have that

$$B_{\mathcal{A},\mathcal{H}} = \left((x_1-x_2)(x_1-x_3)(x_2-x_3) \right) \subset \left((x_1-x_2)(x_1-x_3) \right) = F_{\mathcal{A},\mathcal{H}} = I_{\mathcal{A}}.$$

The ideal $F_{\mathcal{A},\mathcal{H}}$ can be characterized algebraically as follows.

Proposition 4.1.3. *The ideal generated by all products of linear forms in $L_{\mathcal{H}}$ that are contained in $I_{\mathcal{A}}$ is equal to $F_{\mathcal{A},\mathcal{H}}$.*

Proof. Consider a product of linear forms $\ell_1 \cdots \ell_q$ such that each $\ell_i \in L_{\mathcal{H}}$. We have that $\ell_1 \cdots \ell_q \in I_{\mathcal{A}}$, if and only if for each $1 \leq i \leq r$ there exists a linear form $\ell_j \in I_i$. $\qquad\square$

4.2. **The embedding.** The next result shows that the vanishing ideal of every arrangement of two subspaces is pl-generated. This is not true for three subspaces, as shown by an example due to Li and Li [LL2]; see Proposition 5.2.1 and Remark 5.2.3 for comments and generalizations.

Proposition 4.2.1. *Let J and J' be two linear ideals. The ideal $J \cap J'$ is generated by products of linear forms.*

Proof. Let $V = k^n$ denote the ambient vector space, and let V_1 and V_2 be the vector subspaces of V defined by J and J', respectively. Write $V_1 = W_1 \oplus (V_1 \cap V_2)$, $V_2 = W_2 \oplus (V_1 \cap V_2)$, and $V = W \oplus W_1 \oplus W_2 \oplus (V_1 \cap V_2)$, for vector spaces $W_1 \subseteq V_1$, $W_2 \subseteq V_2$, and $W \subseteq V$. Let $\{s_1, \ldots, s_q\}$ be a basis for W, $\{e_1, \ldots, e_b\}$ be a basis for W_1, $\{h_1, \ldots, h_a\}$ be a basis for W_2, and $\{t_1, \ldots, t_c\}$ be a basis for $V_1 \cap V_2$.

Clearly, $J = (s_1, \ldots, s_q, h_1, \ldots, h_a)$ and $J' = (s_1, \ldots, s_q, e_1, \ldots, e_b)$. Since J and J' are monomial ideals, their intersection is generated by $s_1, \ldots, s_q$ and all elements of the form $e_i h_j$. $\qquad\square$

However, even for an arrangement of two subspaces, one can choose a poor embedding into a hyperplane arrangement from which one cannot readily detect if $I_{\mathcal{A}}$ is generated by products of linear forms:

Example 4.2.2 (D. Kozlov)**.** Suppose that the characteristic of k is not equal to 2. Consider the subspace arrangement $\mathcal{A}$ that consists of the two subspaces $\{x_1 - x_2 = 0, \ x_1 + x_2 = 0\}$ and $\{x_1 - x_3 = 0, \ x_1 + x_3 = 0\}$. Take the hyperplane arrangement $\mathcal{H}$ consisting of the hyperplanes $\{x_1 - x_2 = 0\}$, $\{x_1 + x_2 = 0\}$, $\{x_1 - x_3 = 0\}$ and

$\{x_1 + x_3 = 0\}$. Then $A_{\mathcal{A},\mathcal{H}} = A^{**}_{\mathcal{A},\mathcal{H}}$. On the other hand,

$$B_{\mathcal{A},\mathcal{H}} = F_{\mathcal{A},\mathcal{H}} = \Big((x_1 - x_2)(x_1 - x_3), \ (x_1 - x_2)(x_1 + x_3),$$
$$(x_1 + x_2)(x_1 - x_3), \ (x_1 + x_2)(x_1 + x_3) \Big)$$
$$= \Big(x_1^2, x_1 x_2, x_1 x_3, x_2 x_3 \Big)$$

is clearly not a reduced ideal, so it is not equal to $I_{\mathcal{A}}$.

By contrast, let us consider the embedding of $\mathcal{A}$ into the coordinate hyperplane arrangement $\mathcal{C}_3$ consisting of the hyperplanes $\{x_1 = 0\}$, $\{x_2 = 0\}$, $\{x_3 = 0\}$. Denote by J and J' the defining ideals of the two subspaces in $\mathcal{A}$. In this case, $\{x_1\}$ is a basis of the k-space $J_1 \cap J'_1$. Furthermore, $\{x_1, x_2\}$ is a basis of the k-space J_1, and $\{x_1, x_3\}$ is a basis of the k-space J'_1. By Proposition 4.2.1, we conclude that

$$I_{\mathcal{A}} = (\ x_1, \ x_2 x_3\ ) = B_{\mathcal{A},\mathcal{C}_3}\,.$$

This can also be seen to follow from Example 3.4.2.

The problem with the first embedding in Example 4.2.2 is that the ideals $B_{\mathcal{A},\mathcal{H}}$ and $F_{\mathcal{A},\mathcal{H}}$ are strictly smaller than the ideal generated by all products of linear forms in $I_{\mathcal{A}}$. We will show that this problem can be avoided if we take an embedding into a larger hyperplane arrangement $\tilde{\mathcal{H}} \supseteq \mathcal{H}$.

Construction 4.2.3. We start with the embedding $\mathcal{A} \subseteq \mathcal{H}$. If necessary, to the atoms $\ell_1, \ldots, \ell_p$ of $L_{\mathcal{H}}$ we add finitely many new atoms to obtain a new larger hyperplane arrangement $\tilde{\mathcal{H}}$ such that for any choice of $1 \leq i_1 < \cdots < i_t \leq r$ there exists a subset of atoms in $L_{\tilde{\mathcal{H}}}$ that forms a basis for the k-space consisting of the linear forms in $I_{i_1} \cap \cdots \cap I_{i_t}$.

The procedure for enlarging $\mathcal{H}$ to $\tilde{\mathcal{H}}$ is clearly finite, since it amounts to adding a finite number of atoms (linear forms) in each of at most 2^r steps. The key observation is that the ideal $F_{\mathcal{A},\tilde{\mathcal{H}}}$ is the largest possible, and is independent of the choice of $\mathcal{H}$ and $\tilde{\mathcal{H}}$, as we now show.

Theorem 4.2.4. *The ideal generated by all products of linear forms in $I_{\mathcal{A}}$ is equal to the $\tilde{\mathcal{H}}$-product ideal $F_{\mathcal{A},\tilde{\mathcal{H}}}$.*

Proof. A product $f = f_1 \cdots f_q$ of linear forms is in $I_{\mathcal{A}}$ if and only if for each $0 \leq i \leq r$ there exists a linear form $f_j \in I_i$. Furthermore, if $f_q \in I_{i_1} \cap \cdots \cap I_{i_t}$, then f_q is a linear combination of the basis elements of the k-space consisting of all linear forms in $I_{i_1} \cap \cdots \cap I_{i_t}$. Hence, the ideal generated by all products of linear forms in $I_{\mathcal{A}}$ is $F_{\mathcal{A},\tilde{\mathcal{H}}}$. $\square$

B. Sturmfels asked if one can check whether $I_{\mathcal{A}}$ is generated by products of linear forms algorithmically. We obtain such an algorithm as an immediate corollary of Theorem 4.2.4. It can be implemented using the computer algebra system MACAULAY 2 [GS]. Note that our algorithm avoids computing radicals, which are very difficult to compute.

Algorithm 4.2.5. A subspace arrangement $\mathcal{A}$ is given.
(1) Compute $I_{\mathcal{A}}$ as the intersection of the linear defining ideals
    of the subspaces in $\mathcal{A}$.
(2) Choose an embedding into a hyperplane arrangement $\mathcal{H}$.
(3) Construct $\tilde{\mathcal{H}}$.
(4) Construct the $\tilde{\mathcal{H}}$-product ideal $F_{\mathcal{A},\tilde{\mathcal{H}}}$.
(5) Check if the ideals $I_{\mathcal{A}}$ and $F_{\mathcal{A},\tilde{\mathcal{H}}}$ have the same Hilbert function.
    If *YES*: $I_{\mathcal{A}}$ is generated by products of linear forms.
    If *NO*: It is not.

5. Arrangements in $\mathbf{P}^2$ and $\mathbf{P}^3$

The results in this section show that in general, an arrangement of points in $\mathbf{P}^2$ or lines in $\mathbf{P}^3$ will not have a pl-generated ideal.

5.1. **Points in $\mathbf{P}^2$.** Let $S = k[x, y, z]$. It is relatively easy to see by direct computation that if $\mathcal{A}$ is any set of r points in $\mathbf{P}^2$ with $r \leq 4$ then $I_{\mathcal{A}}$ is pl-generated. The possible configurations can be organized according to the maximum number of collinear points. We leave the computation aside.

What happens when $r > 4$? Recall that a set $\mathcal{A}$ of r points in $\mathbf{P}^2$ is *linearly general* if no three are collinear and that a set of r points is *generic* if $\dim_k(S/I_{\mathcal{A}})_t = \min\{r, \binom{t+2}{2}\}$. Five points in $\mathbf{P}^2$ in linearly general position lie on a unique irreducible conic, so their ideal cannot be pl-generated. However, 6 generic points in linearly general position do not lie on a conic, and we show in Proposition 5.1.3 that the ideal of such an arrangement of points is pl-generated. For $r > 6$, Proposition 5.1.4 shows that the ideal of r linearly general points in $\mathbf{P}^2$ is not pl-generated.

It would be interesting to find a characterization of all sets of points in $\mathbf{P}^2$ whose ideals are pl-generated. In Proposition 5.1.6 we give an example of a constraint that one can impose on the geometry of the points that forces their ideal to be pl-generated.

We begin by recalling some information about the ideals of points in $\mathbf{P}^2$. A good reference for these results, which we will cite without proof, is Chapter 3 of [Ei].

The ideal of a finite set of points $\mathcal{A}$ in $\mathbf{P}^2$ has a very beautiful description via the Hilbert-Burch Theorem. Let $S(-d)$ denote the polynomial ring S with degrees shifted so that it is generated in degree d, i.e., the degree m piece is $S(-d)_m = S_{m-d}$. The Hilbert-Burch Theorem says that the ideal $I_{\mathcal{A}}$ is minimally generated by a nonzerodivisor α times the maximal minors of a $(t+1) \times t$ matrix M that can be viewed as a map in the following short exact sequence:

$$0 \longrightarrow \bigoplus_{i=1}^{t} S(-b_i) \xrightarrow{M} \bigoplus_{i=1}^{t+1} S(-a_i) \longrightarrow I_{\mathcal{A}} \longrightarrow 0,$$

where the a_i are the degrees of the elements in a minimal system of generators of $I_{\mathcal{A}}$ and the b_i are the degrees of the elements in a minimal system of generators of the syzygies on the generators of $I_{\mathcal{A}}$.

All of the numerical information associated to $I_{\mathcal{A}}$ is encoded in the degrees of the entries along the two main diagonals of M. Let e_i denote the degree of the (i,i) entry of M and let f_j denote the degree of the $(j, j+1)$ entry of M. The following theorem collects some of the relationships between the numbers we have defined (see Proposition 3.8 in [Ei], for a proof of (1), (2), and (3)).

Theorem 5.1.1. *Assume that $a_1 \geq a_2 \geq \cdots \geq a_{t+1}$ and $b_1 \geq b_2 \geq \cdots \geq b_t$. The following properties hold:*

(1) $e_i, f_i \geq 1$
(2) $a_i = \sum_{j<i} e_j + \sum_{j \geq i} f_j$
(3) $b_i = a_i + e_i$
(4) $\deg \mathcal{A} = \sum_{i \leq j} e_i f_j$ *(Ciliberto-Geramita-Orecchia* [CGO]*).*

We will need the following corollary of the Hilbert-Burch Theorem.

Corollary 5.1.2 (Burch). *If a finite set of points in $\mathbf{P}^2$ lies on a curve of degree d then the ideal of the points can be generated by $d+1$ elements.*

Programs in MACAULAY 2 [GS], one of which was written by D. Eisenbud, suggested that the ideal of six randomly chosen points in $\mathbf{P}^2$ is pl-generated, motivating the following theorem.

Proposition 5.1.3. *If $\mathcal{A}$ is a set of 6 generic and linearly general points in $\mathbf{P}^2$, then $I_{\mathcal{A}}$ is pl-generated.*

Proof. First we will show that $I_{\mathcal{A}}$ must be generated by 4 linearly independent cubics. Then we will construct 4 degree 3 products of linear forms that vanish on $\mathcal{A}$ and are linearly independent.

Since six generic points impose six independent conditions on cubics, and the space of cubics in three variables has dimension 10, we see that

there are precisely 4 linearly independent cubics in $I_\mathcal{A}$. If the points are chosen generically, they will not all lie on a line or a conic. Thus, there are no elements in $I_\mathcal{A}$ of degree ≤ 2. By Corollary 5.1.2, $I_\mathcal{A}$ requires at most 4 generators. Therefore, we see that the 4 cubics in $I_\mathcal{A}$ generate the ideal.

We construct 4 degree 3 forms vanishing on $\mathcal{A}$. Label the points $p_1, \ldots, p_6$ and let $L_{i,j}$ denote the line joining p_i to p_j. Since the points are linearly general, the set of all $L_{i,j}$ with $i < j$ consists of distinct lines. Define cubics

$$Q_1 = L_{1,2} \cdot L_{3,4} \cdot L_{5,6}, \; Q_2 = L_{1,2} \cdot L_{3,5} \cdot L_{4,6},$$

$$Q_3 = L_{1,5} \cdot L_{2,6} \cdot L_{3,4}, \; Q_4 = L_{1,3} \cdot L_{2,6} \cdot L_{4,5}.$$

If they were linearly dependent, then we could find $a, b, c, d \in k$, not all zero, such that the equation

$$aQ_1 + bQ_2 = cQ_3 + dQ_4$$

would be satisfied. But then $L_{1,2}$ divides the lefthand side, so it must also divide the righthand side. The righthand side is also divisible by $L_{2,6}$, so if it is nonzero, it factors as a product of 3 linear forms. However, the third form would have to vanish on p_3, p_4, and p_5, which contradicts our assumption that the points are in linearly general position. We see that $a = b = c = d = 0$ and conclude that the 4 cubics generate $I_\mathcal{A}$. $\qquad\square$

For $r > 6$ an elementary dimension count shows that $I_\mathcal{A}$ cannot be pl-generated if $\mathcal{A}$ consists of r points in linearly general position.

Proposition 5.1.4. *Let $\mathcal{A}$ be an arrangement of $r > 6$ points in $\mathbf{P}^2$ in linearly general position. Then $I_\mathcal{A}$ is not pl-generated.*

Proof. Note that any linear form defines a line in $\mathbf{P}^2$ that contains at most two points of $\mathcal{A}$. Thus, the minimum degree of a product of homogeneous linear forms that vanishes on $\mathcal{A}$ is $\lceil r/2 \rceil$. Hence, we are done if we can show that there must be a form of degree less than $\lceil r/2 \rceil$ in $I_\mathcal{A}$. This follows from the fact that

$$r < \binom{\lceil r/2 \rceil - 1 + 2}{2} = \binom{\lceil r/2 \rceil + 1}{2}$$

if $r > 6$. $\qquad\square$

Remark 5.1.5. An analogous argument shows that for $r \gg 0$, the ideal of a linearly general arrangement of r points in $\mathbf{P}^q$ cannot be pl-generated.

The following proposition gives an example of hypotheses on the geometry of the points that imply that $I_\mathcal{A}$ is pl-generated.

Proposition 5.1.6. *If $\mathcal{A}$ is a set of r points contained in a union of 2 lines then $I_\mathcal{A}$ is pl-generated.*

Proof. As mentioned at the beginning of the section, any set of 4 points in $\mathbf{P}^2$ is pl-generated. So we may assume that $r > 4$. If there exist lines L_1 and L_2 containing $\mathcal{A}$, and $L_1 \cap L_2$ is a point of $\mathcal{A}$, then we can find products of linear forms generating $I_\mathcal{A}$ via a construction of Geramita, Gregory and Roberts [GGR], discussed in Chapter 3 of [Ei].

Otherwise, we may assume that the conic $L_1 \cup L_2$ is unique and that L_1 contains points $p_1, \ldots, p_{r_1} \in \mathcal{A}$ and L_2 contains $q_1, \ldots, q_{r_2} \in \mathcal{A}$ with $r_1 \geq r_2$. For $i = 1, \ldots, r_2$, let h_i be a linear form defining the line joining p_i to q_i. For $i = r_{2+1}, \ldots, r_1$, pick any line through p_i not equal to L_1 and let h_i be its defining equation.

Since $L_1 \cap L_2 \notin \mathcal{A}$, the points are a complete intersection if $r_1 = r_2$, and $I_\mathcal{A} = (L_1 \cdot L_2, h_1 \cdots h_{r_1})$. If $r_1 > r_2$, then using Corollary 5.1.2 and Theorem 5.1.1 we see that $I_\mathcal{A}$ must be minimally generated by the conic $L_1 \cdot L_2$ plus generators of degrees $r_2 + 1$ and r_1. (See also Exercises 3.4–3.7 in [Ei].) Therefore,

$$I_\mathcal{A} = (L_1 \cdot L_2, \; L_1 \cdot h_1 \cdots h_{r_2}, h_1 \cdots h_{r_1}).$$

$\square$

Arrangements of points in $\mathbf{P}^2$ whose ideals are pl-generated also appear in §2 of [GM], and results in [GGR] show that for every Hilbert function of points in $\mathbf{P}^2$ there exists a finite set of points whose ideal is pl-generated having that Hilbert function, as long as k is infinite. In fact, as Theorem 3.13 in [Ei] shows, one can specify the e_i and f_j appearing in Theorem 5.1.1.

5.2. **Lines in $\mathbf{P}^3$.** In this section we will explore when the ideal of an arrangement of lines in $\mathbf{P}^3$ is pl-generated.

It is easy to construct line arrangements whose defining ideals are not pl-generated. Recall that $\mathbf{P}^1 \times \mathbf{P}^1$ can be embedded into $\mathbf{P}^3$ as an irreducible quadric surface Z. Then, over an infinite field, Z has two infinite rulings of disjoint lines $\{X_\alpha\}$ and $\{Y_\beta\}$. Let $\mathcal{A}$ be a line arrangement consisting of r distinct lines from among the set $\{X_\alpha\}$. Since each pair of distinct lines in $\mathcal{A}$ is disjoint, no two lines are contained in a hyperplane. Therefore, the minimum degree of a product of homogeneous linear forms that vanishes on $\mathcal{A}$ is r. Thus, if $r > 2$, it is clear that $I_\mathcal{A}$ cannot be pl-generated.

More generally, we have the following result.

Proposition 5.2.1. *If $\mathcal{A}$ is any collection of $r > 2$ disjoint lines in $\mathbf{P}^3$, then $I_{\mathcal{A}}$ cannot be pl-generated.*

Proof. Since any three skew lines in $\mathbf{P}^3$ lie on an irreducible quadric surface (see [Har], Ex. 2.12.), it follows that there is a form of degree $2\lfloor r/3 \rfloor + a$ in $I_{\mathcal{A}}$, where $a \equiv r \pmod 3$. But no pair of the lines is contained in a hyperplane, so the minimum degree of a product of linear forms vanishing on the r lines is r. $\qquad\square$

Remark 5.2.2. Since one does not expect lines in $\mathbf{P}^3$ to meet, Proposition 5.2.1 implies that in practice, lines in $\mathbf{P}^3$ picked at random will not have a pl-generated ideal.

Remark 5.2.3. Proposition 5.2.1 is a generalization of the example given in [LL2] of three subspaces whose ideal is not pl-generated; the subspaces given there are in fact three skew lines in $\mathbf{P}^3$. One can generalize the statement further to show that the ideal of r skew $(k-1)$-planes in $\mathbf{P}^{2k-1}$ is not pl-generated when $r \gg 0$. Three pairwise disjoint $(k-1)$-planes lie on a variety defined by quadrics which is projectively equivalent to a Segre variety. (See again [Har], Ex. 2.12.) However, no two of the $(k-1)$-planes can lie in a hyperplane. Using products of quadrics, when r is large we can find a form of degree $< r$ that vanishes on the arrangement.

We close this section with an example showing that there are line arrangements in $\mathbf{P}^3$ that cannot be scheme-theoretically defined by any pl-generated ideal.

Recall that the *saturation* of a homogeneous ideal I in $k[x_1, \ldots x_n]$ is defined to be

$$\{ f \in k[x_1, \ldots, x_n] \mid f \cdot (x_1, \ldots, x_n)^d \subseteq I \text{ for } d \gg 0 \}.$$

The saturation of an ideal I is the largest ideal defining the projective subscheme defined by I, and ideals with distinct saturations define distinct schemes.

Let $\mathcal{A}$ be an arrangement of lines in $\mathbf{P}^3$. Construct an embedding of $\mathcal{A}$ into a hyperplane arrangement $\tilde{\mathcal{H}}$ as in Construction 4.2.3. Theorem 4.2.4 states that $F_{\mathcal{A}, \tilde{\mathcal{H}}}$ is the ideal generated by all products of linear forms in $I_{\mathcal{A}}$. The following example shows that it may be the case that the three ideals $I_1 \cdots I_r$, $F_{\mathcal{A}, \tilde{\mathcal{H}}}$, and $I_{\mathcal{A}}$ define three different schemes. (Thanks to D. Eisenbud and R. Lazarsfeld for suggesting to investigate cones of subspaces.)

Proposition 5.2.4. *Let $k[w, x, y, z]$ be the coordinate ring of $\mathbf{P}^3$. Let X be a set of five points in $\mathbf{P}^2$ in linearly general position, and let*

$I_1, \ldots, I_5 \subseteq k[x, y, z]$ *be their defining ideals. Let* $\tilde{I}_i = I_i \cdot k[w, x, y, z]$, *and define* $I_\mathcal{A} = \tilde{I}_1 \cap \cdots \cap \tilde{I}_5$, *so that* $\mathcal{A}$ *is a cone over* X. *Then* $\tilde{I}_1 \cdots \tilde{I}_5$, $F_{\mathcal{A}, \tilde{\mathcal{H}}}$, *and* $I_\mathcal{A}$ *are saturated and are all different.*

Proof. The ideals $\tilde{I}_1 \cdots \tilde{I}_5$, $F_{\mathcal{A}, \tilde{\mathcal{H}}}$, and $I_\mathcal{A}$ are all generated by polynomials in x, y, z and are hence saturated as ideals in $k[w, x, y, z]$.

Since each pair of lines lies in a hyperplane, $F_{\mathcal{A}, \tilde{\mathcal{H}}}$ contains elements of degree three. This shows that it cannot be equal to $\tilde{I}_1 \cdots \tilde{I}_5$, which contains only elements of degree ≥ 5.

Additionally, $F_{\mathcal{A}, \tilde{\mathcal{H}}}$ cannot be equal to $I_\mathcal{A}$ because $I_\mathcal{A}$ contains the equation of the cone over the unique conic determined by the points in X, but $F_{\mathcal{A}, \tilde{\mathcal{H}}}$ contains only forms of degree ≥ 3. $\qquad\square$

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DEPARTMENT OF MATHEMATICS, ROYAL INSTITUTE OF TECHNOLOGY, S-100 44 STOCKHOLM, SWEDEN
  E-mail address: `bjorner@math.kth.se`

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853, USA
  E-mail address: `irena@math.cornell.edu`

DEPARTMENT OF MATHEMATICS AND STATISTICS, 415 A CLAPP LAB, MOUNT HOLYOKE COLLEGE, SOUTH HADLEY, MA 01075, USA
  E-mail address: `jsidman@mtholyoke.edu`